1 #467 Approximate Solution of a Boundary Value Problem Using the Collocation Method in Mathematica

Problem 1: Using the collocation method, find an approximate solution of the equation

$$y'' + (1 + x^2)y + 1 = 0, \tag{1}$$

with the boundary conditions

$$y(-1) = y(1) = 0 \tag{2}$$

Solution:

The given equation has the following form

$$L(u(x)) = f(x), \tag{3}$$

where L is differential operator. We choose an approximate solution

$$y_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \tag{4}$$

where c_k are unknown coefficients; $\phi_k(x)$ are basis functions (polynomials, trigonometric functions, splines, etc.) From the form of the equation and the boundary conditions, one can conclude that the solution of the problem is an even function. Therefore, we choose the following polynomials as basis functions:

$$u_0(x) = 0, \qquad u_1(x) = 1 - x^2, \qquad u_2(x) = x^2(1 - x^2), \tag{5}$$

It is easy to see that the boundary conditions are satisfied for these functions.

We will seek the solution in the form

$$y = c_1(1 - x^2) + c_2 x^2(1 - x^2) \tag{6}$$

As collocation points, we take

$$x_0 = 0, \qquad x_1 = \frac{1}{2} \tag{7}$$

$$\begin{aligned} R(x) &= L(y_N(x)) - f(x) \\ &= y_N''(x) + (1 + x^2)y_N(x) + 1 \\ &= -2c_1 + c_2(2 - 12x^2) + (1 + x^2)\big[c_1(1 - x^2) + c_2(x^2 - x^4)\big] + 1 \\ &= 1 - c_1(1 + x^4) + c_2(2 - 11x^2 - x^6) \end{aligned} \tag{8}$$

Substituting $x_0 = 0, \quad x_1 = \frac{1}{2}$, we obtain the system

$$\begin{cases} 1 - c_1 + 2c_2 = 0, \\ 1 - \frac{17}{16}c_1 - \frac{49}{64}c_2 = 0. \end{cases} \tag{9}$$

Solving it, we find

$$c_1 = 0.957, \qquad c_2 = -0.022 \tag{10}$$

Therefore, the approximate solution has the form

$$y \approx 0.957(1 - x^2) - 0.022x^2(1 - x^2). \tag{11}$$

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