1 Solution of a boundary value problem in double trigonometric series.

Problem 1: Determine the deflection in the center of a rectangular plate hinged at the edges loaded by the load uniformly distributed over the surface of the plate.

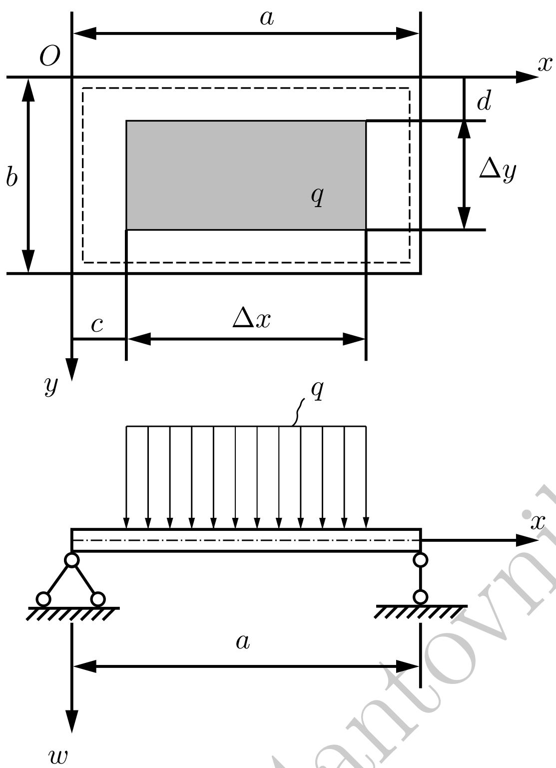

Figure 1: A plate loaded with a load evenly distributed over a rectangular area.

Solution: The solution to the two-dimensional problem for a heterogeneous linear partial differential equation can be found by separating variables using double trigonometric series. In this case, as well as when using single series, the differential operators of the equation and boundary conditions should be even multiplicity. To study the use of double trigonometric series is convenient to consider using the example of solving the problem of bending a thin plate of rectangular shape (Navier's solution). It is required to determine the deflections $w(x, y)$ of a thin plate shown in Figure 2, the bending of which under an arbitrary load $q(x, y)$ is described by the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D}. \tag{1}$$

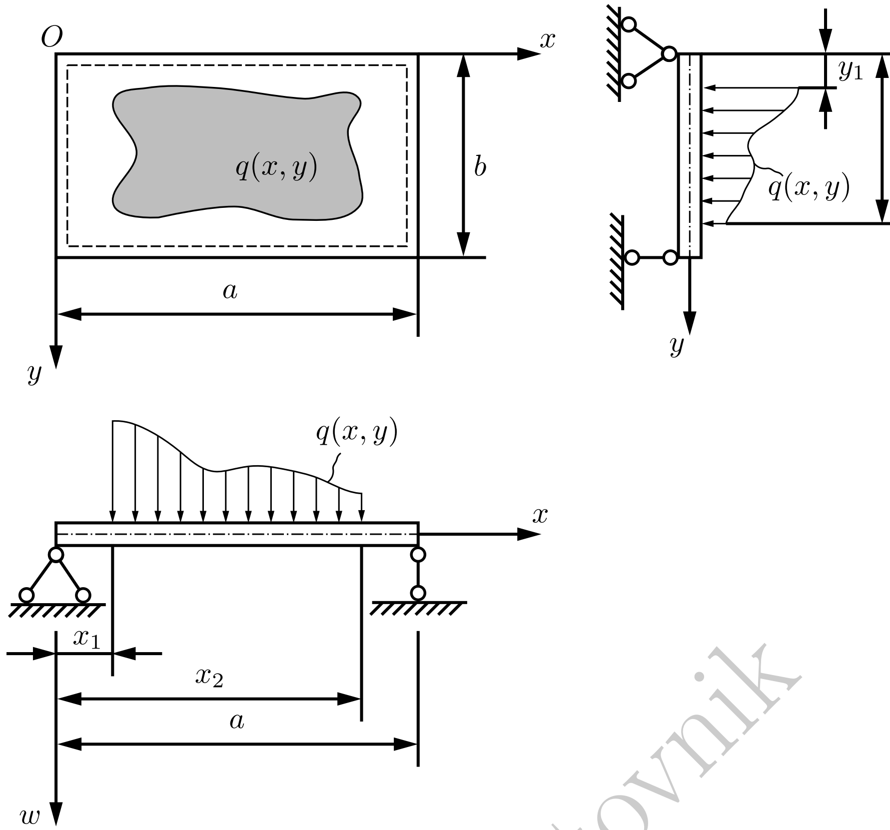

Figure 2: A rectangular plate under the action of an arbitrary load.

The boundary conditions for the plate with hinged support of all edges have the following form

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = 0,\ w\Big|_{\substack{y=0\\y=b}} = 0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{y=0\\y=b}} = 0. \tag{2}$$

The solution to the boundary value problem, Eqs. (1) and (2), is represented as a double trigonometric series

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{3}$$

This solution satisfies boundary conditions, Eq. (2). It satisfies Eq. (1) at some values of constants A_{mn} which are defined by substituting Eq. (3) into Eq. (1):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \frac{q(x, y)}{D}. \tag{4}$$

Thus, the left side of the original equation (1) is represents the Fourier series along the sinus functions. Similarly, one can represent the right part, that is, the load function

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{5}$$

This expression is substituted into Eq. (4)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{6}$$

Equating the coefficients with the same trigonometric functions in the right and left parts of the obtained equality, we get

$$C_{mn} = D\pi^4 A_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2. \tag{7}$$

On the other hand, the formula for determining the Fourier series coefficients for $q(x, y)$ is as follows

$$C_{mn} = \frac{4}{ab} \int_{x_1}^{x_2} \int_{y_1}^{y_2} q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy, \tag{8}$$

where for the integral can be given the designation

$$K_{mn} = \frac{4}{ab} \int_{x_1}^{x_2} \int_{y_1}^{y_2} q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy. \tag{9}$$

Then, we obtain

$$C_{mn} = \frac{4}{ab} K_{mn}. \tag{10}$$

With this expression in mind, for C_{mn}, a formula for determining the values of constants A_{mn} can be derived from Eq. (8)

$$A_{mn} = \frac{4K_{mn}}{abD\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}, \tag{11}$$

and The sought solution of Eq. (3) takes the form

$$w(x, y) = \frac{4}{abD\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{K_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{12}$$

After determining the deflections w, the bending and torquing moments and the transverse forces in the plate are found according to the known equations of the thin plate theory. The double integral is relatively easy to calculate for a particularly given load. For example, for a load evenly distributed over some rectangular region with sides parallel to the x and y axes we have

$$q = \text{const}, \; x_1 = c, \; x_2 = c + \Delta x, \; y_1 = d, \; y_2 = d + \Delta y, \tag{13}$$

and from Eq. (9) we have

$$K_{mn} = q \int_{c}^{c+\Delta x} \sin \frac{m\pi x}{a} \, dx \int_{d}^{d+\Delta y} \sin \frac{n\pi y}{b} \, dy = \frac{qab}{mn\pi^2} \cos \frac{m\pi x}{a} \bigg|_{c}^{c+\Delta x} \cos \frac{n\pi y}{b} \bigg|_{d}^{d+\Delta y} \tag{14}$$

After substituting the limits of integration and trigonometric transformations, we get

$$K_{mn} = \frac{4qab}{mn\pi^2} \left(\sin \frac{m\pi \Delta x}{2a} \sin \frac{m\pi}{a} \left(c + \frac{\Delta x}{2} \right) \right) \left(\sin \frac{n\pi \Delta y}{2b} \sin \frac{n\pi}{b} \left(d + \frac{\Delta y}{2} \right) \right). \tag{15}$$

For the given problem we have

$$c = d = 0, \; \Delta x = a, \; \Delta y = b. \tag{16}$$

Using Eq. (15), we obtain

$$K_{mn} = \frac{4qab}{mn\pi^2} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}. \tag{17}$$

We get that

$$\sin\frac{m\pi}{2}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } m \text{ or } n \text{ is even,} \\ 1 & \text{if } m \text{ and } n \text{ are odd.} \end{cases} \tag{18}$$

Substituting K_{mn} with $m, n = 1, 3, \dots$ into Eq. (12), we obtain

$$w(x, y) = \frac{16qa^4}{D\pi^6} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(m^2 + n^2\frac{a^2}{b^2}\right)^2}. \tag{19}$$

In order to get a reasonably accurate result, only one member of the double series can be stored in this formula ($m = n = 1$). In this case, at $a = b$ in the center of the plate ($x = a/2, y = b/2$), the deflection is

$$w_{max} = \frac{4qa^4}{D\pi^6}. \tag{20}$$

Saving two terms of the series ($m, n = 1, 3$) adjusts this result not exceeding 3%.

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