

Problem 1: A beam of variable cross-section is stretched by its own weight and compressed by a force P applied at the end of the beam as shown in Figure 1. Determine the magnitude of the force at which the displacement of the section A-A will be zero. The material density is ρ , modulus of elasticity is E , and the cross-sectional area is $F(z)$.

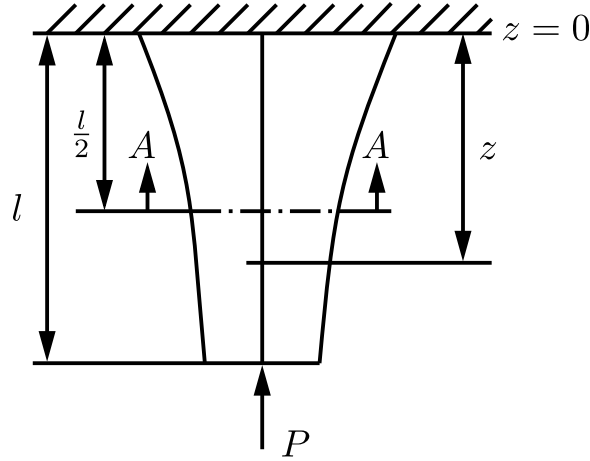


Figure 1: A beam of variable cross-section.

Solution: Let us determine the magnitude of the normal force $N(z)$ in an arbitrary section z .

$$N(z) = \int_z^l p(z) dz - P. \quad (1)$$

Since

$$p(z) = \rho F(z) \quad (2)$$

then

$$N(z) = \rho \int_z^l F(z) dz - P. \quad (3)$$

and

$$\sigma(z) = \frac{N(z)}{F(z)} = \frac{\rho}{F(z)} \int_z^l F(z_1) dz_1 - \frac{P}{F(z)}. \quad (4)$$

We know that the displacement $w(z)$ of an arbitrary section z is equal to

$$w(z) = \int_0^z \varepsilon(z_1) dz_1 = \frac{1}{E} \int_0^z \sigma(z_1) dz_1 = \frac{1}{E} \int_0^z \frac{\rho \int_z^l F(z_1) dz_1 - P}{F(z_1)} dz_1 \quad (5)$$

or

$$w(z) = \frac{1}{E} \int_0^z \frac{\rho \int_z^l F(z_1) dz_1}{F(z_1)} dz - \frac{P}{E} \int_0^z \frac{dz_1}{F(z_1)} \quad (6)$$

Now we can find force P at which the displacement of the section A-A with the coordinate z^* is zero.

$$w(z^*) = \frac{1}{E} \int_0^{z^*} \frac{\rho \int_z^l F(z_1) dz_1}{F(z_1)} dz - \frac{P}{E} \int_0^{z^*} \frac{dz_1}{F(z_1)} = 0 \quad (7)$$

$$\frac{1}{E} \int_0^{z^*} \frac{\rho \int_z^l F(z_1) dz_1}{F(z_1)} = \frac{P}{E} \int_0^{z^*} \frac{dz_1}{F(z_1)} \quad (8)$$

$$P = \rho \frac{\int_0^{z^*} \frac{\int_z^l F(z_1) dz_1}{F(z_1)} dz}{\int_0^{z^*} \frac{dz_1}{F(z_1)}} \quad (9)$$

If the beam has a constant cross section, that is, $F(z) = F$, then at $z^* = \frac{l}{2}$, we have

$$\begin{aligned} P &= \rho F \frac{\int_0^{z^*} \left[\int_z^l 1 dz_1 \right] dz}{\int_0^{z^*} 1 dz_1} \Rightarrow \\ P &= \rho F \frac{\int_0^{z^*} (l - z) dz}{\int_0^{z^*} 1 dz_1} = \rho F \frac{\left[lz^* - \frac{(z^*)^2}{2} \right]}{z^*} = \rho F \left[l - \frac{z^*}{2} \right] \Rightarrow \\ P &= \frac{3}{4} \rho Fl \end{aligned} \quad (10)$$

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